5.22 A long cylindrical conductor whose axis is coincident with the z-axis has a radius a and carries a current characterized by a current density $\mathbf{J} = \hat{\mathbf{z}} J_0/r$, where J_0 is a constant and r is the radial distance from the cylinder's axis. Obtain an expression for the magnetic field $\mathbf{H}$ for

(a) $0 \le r \le a$

(b) $r > a$

Solution: This problem is very similar to Example 5-5.

(a) For $0 \le r_1 \le a$, the total current flowing within the contour C_1 is

$$I_1 = \iint \vec{J} \cdot d\vec{s} = \int_{\phi=0}^{2\pi} \int_{r=0}^{r_1} \left(\frac{\hat{\mathbf{z}} J_0}{r} \right) \cdot (\hat{\mathbf{z}} r \, dr \, d\phi) = 2\pi \int_{r=0}^{r_1} J_0 \, dr = 2\pi r_1 J_0.$$

Therefore, since $I_1 = 2\pi r_1 H_1$, $H_1 = J_0$ within the wire and $\vec{H}_1 = \hat{\boldsymbol{\phi}} J_0$.

(b) For $r \ge a$, the total current flowing within the contour is the total current flowing within the wire:

$$I = \iint \vec{J} \cdot d\vec{s} = \int_{\phi=0}^{2\pi} \int_{r=0}^{a} \left(\frac{\hat{\mathbf{z}} J_0}{r} \right) \cdot (\hat{\mathbf{z}} r \, dr \, d\phi) = 2\pi \int_{r=0}^{a} J_0 \, dr = 2\pi a J_0.$$

Therefore, since $I = 2\pi r H_2$, $H_2 = J_0 a/r$ within the wire and $\vec{H}_2 = \hat{\boldsymbol{\phi}} J_0 (a/r)$.