6.27 A Hertzian dipole is a short conducting wire carrying an approximately constant current over its length l. If such a dipole is placed along the z-axis with its midpoint at the origin, and if the current flowing through it is $i(t) = I_0 \cos \omega t$, find the following:

(a) The retarded vector potential $\widetilde{\mathbf{A}}(R, \theta, \phi)$ at an observation point $Q(R, \theta, \phi)$ in a spherical coordinate system.

(b) The magnetic field phasor $\widetilde{\mathbf{H}}(R, \theta, \phi)$.

Assume l to be sufficiently small so that the observation point is approximately equidistant to all points on the dipole; that is, assume $R' \simeq R$.

Solution:

(a) In phasor form, the current is given by $\widetilde{I} = I_0$. Explicitly writing the volume integral in Eq. (6.84) as a double integral over the wire cross section and a single integral over its length,

$$\widetilde{\vec{A}} = \frac{\mu}{4\pi} \int_{-l/2}^{l/2} \iint_s \frac{\widetilde{\vec{J}}\left(\vec{R}_{\mathrm{i}}\right) \exp -jkR'}{R'} \, ds \, dz,$$

where s is the wire cross section. The wire is infinitesimally thin, so that R' is not a function of x or y and the integration over the cross section of the wire applies only to the current density. Recognizing that $\widetilde{J} = \hat{\mathbf{z}} I_0 / s$, and employing the relation $R' \approx R$,

$$\widetilde{\vec{A}} = \hat{\mathbf{z}} \frac{\mu I_0}{4\pi} \int_{-l/2}^{l/2} \frac{\exp -jkR'}{R'} \, dz \approx \hat{\mathbf{z}} \frac{\mu I_0}{4\pi} \int_{-l/2}^{l/2} \frac{\exp -jkR}{R} \, dz = \hat{\mathbf{z}} \frac{\mu I_0 l}{4\pi R} \exp -jkR.$$

In spherical coordinates, $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta$, and therefore

$$\widetilde{\vec{A}} = \left(\hat{\mathbf{R}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta \right) \frac{\mu I_0 l}{4\pi R} \exp -jkR.$$

(b) From Eq. (6.85),

$$\begin{aligned}
\widetilde{\vec{H}} = \frac{1}{\mu} \nabla \times \widetilde{\vec{A}} &= \frac{I_0 l}{4\pi} \nabla \times \left[\left(\hat{\mathbf{R}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta \right) \frac{\exp -jkR}{R} \right] \\
&= \frac{I_0 l}{4\pi} \hat{\boldsymbol{\phi}} \frac{1}{R} \left(\frac{\partial}{\partial R} (-\sin \theta \exp -jkR) - \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\exp -jkR}{R} \right) \right) \\
&= \hat{\boldsymbol{\phi}} \frac{I_0 l \sin \theta \exp -jkR}{4\pi R} \left(jk + \frac{1}{R} \right).
\end{aligned}$$