2.23 A load with impedance $Z_L = (50 - j50)\ \Omega$ is to be connected to a lossless transmission line with characteristic impedance Z_0, with Z_0 chosen such that the standing-wave ratio is the smallest possible. What should Z_0 be?

Solution: Since S is monotonic with $|\Gamma|$ (i.e., a plot of S vs. $|\Gamma|$ is always increasing), the value of Z_0 which gives the minimum possible S also gives the minimum possible $|\Gamma|$, and, for that matter, the minimum possible $|\Gamma|^2$. A necessary condition for a minimum is that its derivative be equal to zero:

$$0 = \frac{\partial}{\partial Z_0}|\Gamma|^2 = \frac{\partial}{\partial Z_0}\frac{|R_L + jX_L - Z_0|^2}{|R_L + jX_L + Z_0|^2}$$

$$= \frac{\partial}{\partial Z_0}\frac{(R_L - Z_0)^2 + X_L^2}{(R_L + Z_0)^2 + X_L^2} = \frac{4R_L\left(Z_0^2 - \left(R_L^2 + X_L^2\right)\right)}{\left((R_L + Z_0)^2 + X_L^2\right)^2}.$$

Therefore, $Z_0^2 = R_L^2 + X_L^2$ or

$$Z_0 = |Z_L| = \sqrt{(50^2 + (-50)^2)} = 70.7\ \Omega.$$

A mathematically precise solution will also demonstrate that this point is a minimum (by calculating the second derivative, for example). Since the endpoints of the range may be local minima or maxima without the derivative being zero there, the endpoints (namely $Z_0 = 0\ \Omega$ and $Z_0 = \infty\ \Omega$) should be checked also.