3.15 A certain plane is described by

$$2x + 3y + 4z = 16.$$

Find the unit vector normal to the surface in the direction away from the origin.

Solution: Procedure:

1. Use the equation for the given plane to find three points, P_1, P_2 and P_3 on the plane.
2. Find vector $\mathbf{A}$ from P_1 to P_2 and vector $\mathbf{B}$ from P_1 to P_3.
3. Cross product of $\mathbf{A}$ and $\mathbf{B}$ gives a vector $\mathbf{C}$ orthogonal to $\mathbf{A}$ and $\mathbf{B}$, and hence to the plane.
4. Check direction of $\hat{\mathbf{c}}$.

Steps:

1. Choose the following three points:

$$\begin{aligned} &P_1 \text{ at } (0,0,4), \\ &P_2 \text{ at } (8,0,0), \\ &P_3 \text{ at } (0,\tfrac{16}{3},0). \end{aligned}$$

2. Vector $\mathbf{A}$ from P_1 to P_2

$$\mathbf{A} = \hat{\mathbf{x}}(8-0) + \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(0-4) = \hat{\mathbf{x}}8 - \hat{\mathbf{z}}4$$

Vector $\mathbf{B}$ from P_1 to P_3

$$\mathbf{B} = \hat{\mathbf{x}}(0-0) + \hat{\mathbf{y}}\left(\frac{16}{3} - 0\right) + \hat{\mathbf{z}}(0-4) = \hat{\mathbf{y}}\frac{16}{3} - \hat{\mathbf{z}}4$$

3.

$$\begin{aligned} \mathbf{C} &= \mathbf{A} \times \mathbf{B} \\ &= \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) + \hat{\mathbf{z}}(A_xB_y - A_yB_x) \\ &= \hat{\mathbf{x}}\left(0\cdot(-4) - (-4)\cdot\frac{16}{3}\right) + \hat{\mathbf{y}}\left((-4)\cdot 0 - 8\cdot(-4)\right) + \hat{\mathbf{z}}\left(8\cdot\frac{16}{3} - 0\cdot 0\right) \\ &= \hat{\mathbf{x}}\frac{64}{3} + \hat{\mathbf{y}}32 + \hat{\mathbf{z}}\frac{128}{3} \end{aligned}$$

Verify that $\mathbf{C}$ is orthogonal to $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A}\cdot\mathbf{C} = \left(8\cdot\frac{64}{3}\right) + (32\cdot 0) + \left(\frac{128}{3}\cdot(-4)\right) = \frac{512}{3} - \frac{512}{3} = 0$$

$$\mathbf{B}\cdot\mathbf{C} = \left(0\cdot\frac{64}{3}\right) + \left(32\cdot\frac{16}{3}\right) + \left(\frac{128}{3}\cdot(-4)\right) = \frac{512}{3} - \frac{512}{3} = 0$$

4. $\mathbf{C} = \hat{\mathbf{x}}\,\frac{64}{3} + \hat{\mathbf{y}}\,32 + \hat{\mathbf{z}}\,\frac{128}{3}$

$$\hat{\mathbf{c}} = \frac{\mathbf{C}}{|\mathbf{C}|} = \frac{\hat{\mathbf{x}}\,\frac{64}{3} + \hat{\mathbf{y}}\,32 + \hat{\mathbf{z}}\,\frac{128}{3}}{\sqrt{\left(\frac{64}{3}\right)^2 + 32^2 + \left(\frac{128}{3}\right)^2}} = \hat{\mathbf{x}}\,0.37 + \hat{\mathbf{y}}\,0.56 + \hat{\mathbf{z}}\,0.74.$$

$\hat{\mathbf{c}}$ points away from the origin as desired.