3.47 For the vector field $\mathbf{E} = \hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z$, verify the divergence theorem for the cylindrical region enclosed by $r = 2,\ z = 0$, and $z = 4$.

Solution:

$$\begin{aligned}
\oint \vec{E}\cdot d\vec{s} &= \int_{r=0}^{2}\int_{\phi=0}^{2\pi}\left((\hat{\mathbf{r}}10\exp{-r} - \hat{\mathbf{z}}3z)\cdot(-\hat{\mathbf{z}}r\,dr\,d\phi)\right)\Big|_{z=0} \\
&\quad + \int_{\phi=0}^{2\pi}\int_{z=0}^{4}\left((\hat{\mathbf{r}}10\exp{-r} - \hat{\mathbf{z}}3z)\cdot(\hat{\mathbf{r}}r\,d\phi\,dz)\right)\Big|_{r=2} \\
&\quad + \int_{r=0}^{2}\int_{\phi=0}^{2\pi}\left((\hat{\mathbf{r}}10\exp{-r} - \hat{\mathbf{z}}3z)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)\right)\Big|_{z=4} \\
&= 0 + \int_{\phi=0}^{2\pi}\int_{z=0}^{4} 10\exp{-2}2\,d\phi\,dz + \int_{r=0}^{2}\int_{\phi=0}^{2\pi} -12r\,dr\,d\phi \\
&= 160\pi\exp{-2} - 48\pi \approx -82.77,
\end{aligned}$$

$$\begin{aligned}
\iiint \nabla\cdot\vec{E}\,d\mathcal{V} &= \int_{z=0}^{4}\int_{r=0}^{2}\int_{\phi=0}^{2\pi}\left(\frac{10\exp{-r}(1-r)}{r} - 3\right) r\,d\phi\,dr\,dz \\
&= 8\pi\int_{r=0}^{2}(10\exp{-r}(1-r) - 3r)\,dr \\
&= 8\pi\left(-10\exp{-r} + 10\exp{-r}(1+r) - \frac{3r^2}{2}\right)\Bigg|_{r=0}^{2} \\
&= 160\pi\exp{-2} - 48\pi \approx -82.77.
\end{aligned}$$