

**3.49** For the vector field  $\mathbf{D} = \hat{\mathbf{R}}3R^2$ , evaluate both sides of the divergence theorem for the region enclosed between the spherical shells defined by  $R = 1$  and  $R = 2$ .

**Solution:** The divergence theorem is given by Eq. (3.98). Evaluating the left hand side:

$$\begin{aligned}\int_{\mathcal{V}} \nabla \cdot \vec{D} d\mathcal{V} &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=1}^2 \left( \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 (3R^2)) \right) R^2 \sin \theta dR d\theta d\phi \\ &= 2\pi (-\cos \theta) \Big|_{\theta=0}^{\pi} (3R^4) \Big|_{R=1}^2 = 180\pi.\end{aligned}$$

The right hand side evaluates to

$$\begin{aligned}\oint_S \vec{D} \cdot d\vec{s} &= \left( \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (\hat{\mathbf{R}}3R^2) \cdot (-\hat{\mathbf{R}}R^2 \sin \theta d\theta d\phi) \right) \Big|_{R=1} \\ &\quad + \left( \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (\hat{\mathbf{R}}3R^2) \cdot (\hat{\mathbf{R}}R^2 \sin \theta d\theta d\phi) \right) \Big|_{R=2} \\ &= -2\pi \int_{\theta=0}^{\pi} 3 \sin \theta d\theta + 2\pi \int_{\theta=0}^{\pi} 48 \sin \theta d\theta = 180\pi.\end{aligned}$$


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