3.53 Repeat Problem 3.52 for the contour shown in Fig. P3.52(b).

Solution: (a)

$$\oint \vec{B}\cdot d\vec{l} = \int_{L_1} \vec{B}\cdot d\vec{l} + \int_{L_2} \vec{B}\cdot d\vec{l} + \int_{L_3} \vec{B}\cdot d\vec{l} + \int_{L_4} \vec{B}\cdot d\vec{l},$$

$$\vec{B}\cdot d\vec{l} = \left(\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi\right)\cdot\left(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz\right) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$

$$\int_{L_1} \vec{B}\cdot d\vec{l} = \left(\int_{r=1}^{2} r\cos\phi\,dr\right)\bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(\tfrac{1}{2}r^2\right)\Big|_{r=1}^{2} + 0 = \frac{3}{2},$$

$$\int_{L_2} \vec{B}\cdot d\vec{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=0}^{\pi/} r\sin\phi\,d\phi\right)\bigg|_{r=2,\ z=0}$$

$$= 0 + (-2\cos\phi)\big|_{\phi=0}^{\pi/2} = 2,$$

$$\int_{L_3} \vec{B}\cdot d\vec{l} = \left(\int_{r=2}^{1} r\cos\phi\,dr\right)\bigg|_{\phi=\pi/2,\ z=0} + \left(\int_{\phi=\pi/2}^{\pi/2} r\sin\phi\,d\phi\right)\bigg|_{z=0} = 0,$$

$$\int_{L_4} \vec{B}\cdot d\vec{l} = \left(\int_{r=1}^{1} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=\pi/2}^{0} r\sin\phi\,d\phi\right)\bigg|_{r=1,\ z=0}$$

$$= 0 + (-\cos\phi)\big|_{\phi=\pi/2}^{0} = -1,$$

$$\oint \vec{B}\cdot d\vec{l} = \frac{3}{2} + 2 + 0 - 1 = \frac{5}{2}.$$

(b)

$$\nabla\times\vec{B} = \nabla\times\left(\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi\right)$$

$$= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}(\sin\phi)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi) - \frac{\partial}{\partial r}0\right)$$

$$+ \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi)) - \frac{\partial}{\partial\phi}(r\cos\phi)\right)$$

$$= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}(\sin\phi + (r\sin\phi)) = \hat{\mathbf{z}}\sin\phi\left(1 + \frac{1}{r}\right),$$

$$\iint \nabla\times\vec{B}\cdot d\vec{s} = \int_{\phi=0}^{\pi/2}\int_{r=1}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)$$

$$= \int_{\phi=0}^{\pi/2}\int_{r=1}^{2} \sin\phi\,(r+1)\,dr\,d\phi$$

$$= \left(\left(-\cos\phi \left(\tfrac{1}{2}r^2 + r\right)\right)\Big|_{r=1}^{2} \right)\Bigg|_{\phi=0}^{\pi/2} = \frac{5}{2}\,.$$