3.54 Verify Stokes's Theorem for the vector field $\mathbf{A} = \hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta$ by evaluating it on the hemisphere of unit radius.

Solution:

$$\mathbf{A} = \hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi.$$

Hence, $A_R = \cos\theta,\ A_\theta = 0,\ A_\phi = \sin\theta$.

$$\begin{aligned}
\nabla\times\mathbf{A} &= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\right) - \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \hat{\boldsymbol{\phi}}\frac{1}{R}\frac{\partial A_R}{\partial\theta} \\
&= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta) - \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial R}(R\sin\theta) - \hat{\boldsymbol{\phi}}\frac{1}{R}\frac{\partial}{\partial\theta}(\cos\theta) \\
&= \hat{\mathbf{R}}\frac{2\cos\theta}{R} - \hat{\boldsymbol{\theta}}\frac{\sin\theta}{R} + \hat{\boldsymbol{\phi}}\frac{\sin\theta}{R}.
\end{aligned}$$

For the hemispherical surface, $d\mathbf{s} = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$.

$$\begin{aligned}
&\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}(\nabla\times\mathbf{A})\cdot d\mathbf{s} \\
&= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left(\frac{\hat{\mathbf{R}}2\cos\theta}{R} - \hat{\boldsymbol{\theta}}\frac{\sin\theta}{R} + \hat{\boldsymbol{\phi}}\frac{\sin\theta}{R}\right)\cdot\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi\bigg|_{R=1} \\
&= 4\pi R\left.\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2}\right|_{R=1} = 2\pi.
\end{aligned}$$

The contour C is the circle in the x–y plane bounding the hemispherical surface.

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_{\phi=0}^{2\pi}(\hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta)\cdot\hat{\boldsymbol{\phi}}R\,d\phi\bigg|_{\substack{\theta=\pi/2\\R=1}} = R\sin\theta\int_0^{2\pi}d\phi\bigg|_{\substack{\theta=\pi/2\\R=1}} = 2\pi.$$