

3.57 Find the Laplacian of the following scalar functions:

(a) $V_1 = 10r^3 \sin 2\phi$

(b) $V_2 = (2/R^2) \cos \theta \sin \phi$

Solution:

(a)

$$\begin{aligned}\nabla^2 V_1 &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V_1}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_1}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \\&= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} (10r^3 \sin 2\phi) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} (10r^3 \sin 2\phi) + 0 \\&= \frac{1}{r} \frac{\partial}{\partial r} (30r^3 \sin 2\phi) - \frac{1}{r^2} (10r^3) 4 \sin 2\phi \\&= 90r \sin 2\phi - 40r \sin 2\phi = 50r \sin 2\phi.\end{aligned}$$

(b)

$$\begin{aligned}\nabla^2 V_2 &= \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V_2}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V_2}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V_2}{\partial \phi^2} \\&= \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial}{\partial R} \left(\frac{2}{R^2} \cos \theta \sin \phi \right) \right) \\&\quad + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \left(\frac{2}{R^2} \cos \theta \sin \phi \right) \right) \\&\quad + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \left(\frac{2}{R^2} \cos \theta \sin \phi \right) \\&= \frac{4}{R^4} \cos \theta \sin \phi - \frac{4}{R^4} \cos \theta \sin \phi - \frac{2}{R^4} \frac{\cos \theta}{\sin^2 \theta} \sin \phi \\&= -\frac{2}{R^4} \frac{\cos \theta \sin \phi}{\sin^2 \theta}.\end{aligned}$$