4.23 Repeat Problem 4.22 for $\mathbf{D} = \hat{\mathbf{x}}xy^3z^3$ (C/m^2).

Solution:

(a) From Eq. (4.26), $\rho_{\text{v}} = \nabla \cdot \mathbf{D} = \dfrac{\partial}{\partial x}(xy^3z^3) = y^3z^3$.

(b) Total charge Q is given by Eq. (4.27):

$$Q = \int_{\mathscr{V}} \nabla \cdot \mathbf{D}\, d\mathscr{V} = \int_{z=0}^{2} \int_{y=0}^{2} \int_{x=0}^{2} y^3z^3\, dx\, dy\, dz = \frac{xy^4z^4}{16}\bigg|_{x=0}^{2}\bigg|_{y=0}^{2}\bigg|_{z=0}^{2} = 32 \text{ C}.$$

(c) Using Gauss' law we have

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}.$$

Note that $\mathbf{D} = \hat{\mathbf{x}}D_x$, so only F_{front} and F_{back} (integration over $\hat{\mathbf{z}}$ surfaces) will contribute to the integral.

$$\begin{aligned} F_{\text{front}} &= \int_{z=0}^{2} \int_{y=0}^{2} (\hat{\mathbf{x}}xy^3z^3)\bigg|_{x=2} \cdot (\hat{\mathbf{x}}\, dy\, dz) \\ &= \int_{z=0}^{2} \int_{y=0}^{2} xy^3z^3\bigg|_{x=2} dy\, dz = \left(2\left(\frac{y^4z^4}{16}\right)\bigg|_{y=0}^{2}\right)\bigg|_{z=0}^{2} = 32\,, \\ F_{\text{back}} &= \int_{z=0}^{2} \int_{y=0}^{2} (\hat{\mathbf{x}}xy^3z^3)\bigg|_{x=0} \cdot (-\hat{\mathbf{x}}\, dy\, dz) = -\int_{z=0}^{2} \int_{y=0}^{2} xy^3z^3\bigg|_{x=0} dy\, dz = 0. \end{aligned}$$

Thus $Q = \oint \mathbf{D} \cdot d\mathbf{s} = 32 + 0 + 0 + 0 + 0 + 0 = 32$ C.