4.49 An infinitely long conducting cylinder of radius a has a surface charge density ρ_s. The cylinder is surrounded by a dielectric medium with $\varepsilon_r = 4$ and contains no free charges. The tangential component of the electric field in the region $r \geq a$ is given by $\mathbf{E}_t = -\hat{\boldsymbol{\phi}} \cos\phi / r^2$. Since a static conductor cannot have any tangential field, this must be cancelled by an externally applied electric field. Find the surface charge density on the conductor.

Solution: Let the conducting cylinder be medium 1 and the surrounding dielectric medium be medium 2. In medium 2,

$$\mathbf{E}_2 = \hat{\mathbf{r}} E_r - \hat{\boldsymbol{\phi}} \frac{1}{r^2} \cos\phi,$$

with E_r, the normal component of $\mathbf{E}_2$, unknown. The surface charge density is related to E_r. To find E_r, we invoke Gauss's law in medium 2:

$$\nabla \cdot \mathbf{D}_2 = 0,$$

or

$$\frac{1}{r} \frac{\partial}{\partial r}(rE_r) + \frac{1}{r} \frac{\partial}{\partial \phi}\left(-\frac{1}{r^2} \cos\phi\right) = 0,$$

which leads to

$$\frac{\partial}{\partial r}(rE_r) = \frac{\partial}{\partial \phi}\left(\frac{1}{r^2} \cos\phi\right) = -\frac{1}{r^2} \sin\phi.$$

Integrating both sides with respect to r,

$$\int \frac{\partial}{\partial r}(rE_r)\, dr = -\sin\phi \int \frac{1}{r^2}\, dr$$

$$rE_r = \frac{1}{r} \sin\phi,$$

or

$$E_r = \frac{1}{r^2} \sin\phi.$$

Hence,

$$\mathbf{E}_2 = \hat{\mathbf{r}} \frac{1}{r^2} \sin\phi.$$

According to Eq. (4.93),

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s,$$

where $\hat{\mathbf{n}}_2$ is the normal to the boundary and points away from medium 1. Hence, $\hat{\mathbf{n}}_2 = \hat{\mathbf{r}}$. Also, $\mathbf{D}_1 = 0$ because the cylinder is a conductor. Consequently,

$$\rho_s = -\hat{\mathbf{r}} \cdot \mathbf{D}_2 \big|_{r=a}$$

$$
\begin{aligned}
&= -\hat{\mathbf{r}} \cdot \varepsilon_2 \mathbf{E}_2 \big|_{r=a} \\
&= -\hat{\mathbf{r}} \cdot \varepsilon_{\mathrm{r}} \varepsilon_0 \left[\hat{\mathbf{r}} \, \frac{1}{r^2} \sin\phi \right] \Bigg|_{r=a} \\
&= -\frac{4\varepsilon_0}{a^2} \sin\phi \qquad (\mathrm{C/m^2}).
\end{aligned}
$$