

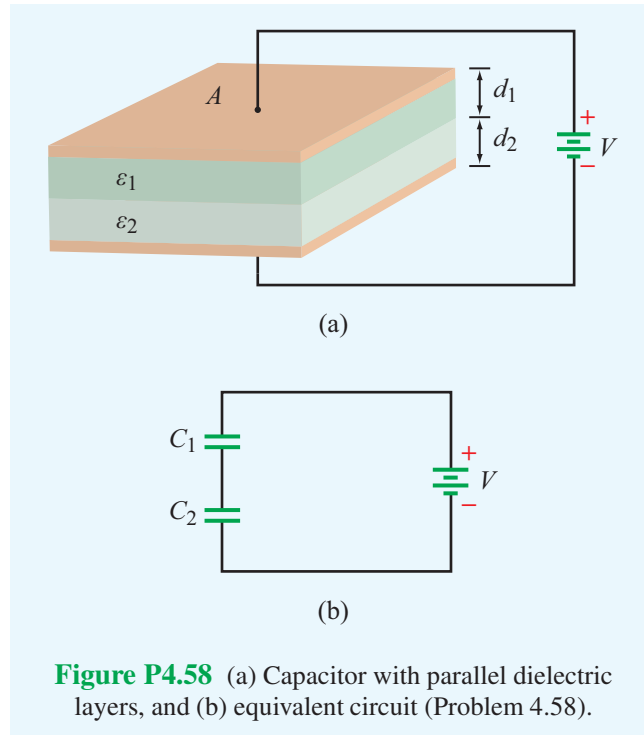
**4.58** The capacitor shown in Fig. P4.58 consists of two parallel dielectric layers. Use energy considerations to show that the equivalent capacitance of the overall capacitor,  $C$ , is equal to the series combination of the capacitances of the individual layers,  $C_1$  and  $C_2$ , namely

$$C = \frac{C_1 C_2}{C_1 + C_2} \quad (22)$$

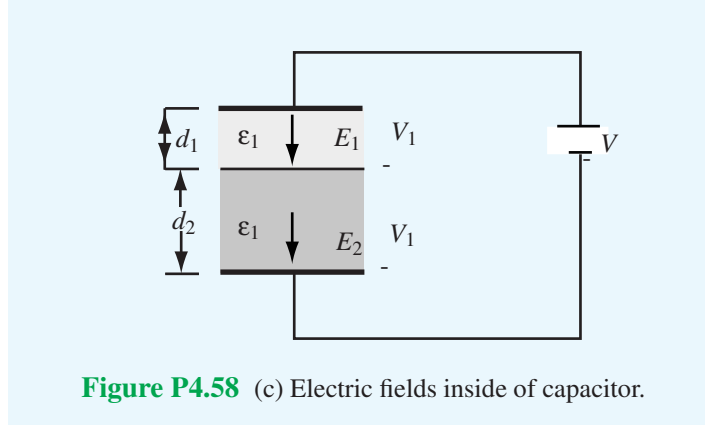
where

$$C_1 = \epsilon_1 \frac{A}{d_1}, \quad C_2 = \epsilon_2 \frac{A}{d_2}$$

- (a) Let  $V_1$  and  $V_2$  be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields  $E_1$  and  $E_2$ ? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for  $E_1$  and  $E_2$  in terms of  $\epsilon_1$ ,  $\epsilon_2$ ,  $V$ , and the indicated dimensions of the capacitor.
- (b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for  $C$ .
- (c) Show that  $C$  is given by Eq. (22).



**Solution:**



(a) If  $V_1$  is the voltage across the top layer and  $V_2$  across the bottom layer, then

$$V = V_1 + V_2,$$

and

$$E_1 = \frac{V_1}{d_1}, \quad E_2 = \frac{V_2}{d_2}.$$

According to boundary conditions, the normal component of  $\mathbf{D}$  is continuous across the boundary (in the absence of surface charge). This means that at the interface between the two dielectric layers,

$$D_{1n} = D_{2n}$$

or

$$\epsilon_1 E_1 = \epsilon_2 E_2.$$

Hence,

$$V = E_1 d_1 + E_2 d_2 = E_1 d_1 + \frac{\epsilon_1 E_1}{\epsilon_2} d_2,$$

which can be solved for  $E_1$ :

$$E_1 = \frac{V}{d_1 + \frac{\epsilon_1}{\epsilon_2} d_2}.$$

Similarly,

$$E_2 = \frac{V}{d_2 + \frac{\epsilon_2}{\epsilon_1} d_1}.$$

(b)

$$W_{e_1} = \frac{1}{2} \varepsilon_1 E_1^2 \cdot \mathcal{V}_1 = \frac{1}{2} \varepsilon_1 \left( \frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2} \right)^2 \cdot A d_1 = \frac{1}{2} V^2 \left[ \frac{\varepsilon_1 \varepsilon_2^2 A d_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} \right],$$

$$W_{e_2} = \frac{1}{2} \varepsilon_2 E_2^2 \cdot \mathcal{V}_2 = \frac{1}{2} \varepsilon_2 \left( \frac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1} \right)^2 \cdot A d_2 = \frac{1}{2} V^2 \left[ \frac{\varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right],$$

$$W_e = W_{e_1} + W_{e_2} = \frac{1}{2} V^2 \left[ \frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right].$$

But  $W_e = \frac{1}{2} C V^2$ , hence,

$$C = \frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \varepsilon_1 \varepsilon_2 A \frac{(\varepsilon_2 d_1 + \varepsilon_1 d_2)}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \frac{\varepsilon_1 \varepsilon_2 A}{\varepsilon_2 d_1 + \varepsilon_1 d_2}.$$

(c) Multiplying numerator and denominator of the expression for  $C$  by  $A/d_1 d_2$ , we have

$$C = \frac{\frac{\varepsilon_1 A}{d_1} \cdot \frac{\varepsilon_2 A}{d_2}}{\frac{\varepsilon_1 A}{d_1} + \frac{\varepsilon_2 A}{d_2}} = \frac{C_1 C_2}{C_1 + C_2},$$

where

$$C_1 = \frac{\varepsilon_1 A}{d_1}, \quad C_2 = \frac{\varepsilon_2 A}{d_2}.$$

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