4.62 With reference to Fig. P4.62, charge Q is located at a distance d above a grounded half-plane located in the x–y plane and at a distance d from another grounded half-plane in the x–z plane. Use the image method to

(a) Establish the magnitudes, polarities, and locations of the images of charge Q with respect to each of the two ground planes (as if each is infinite in extent).

(b) Find the electric potential and electric field at an arbitrary point $P = (0, y, z)$.

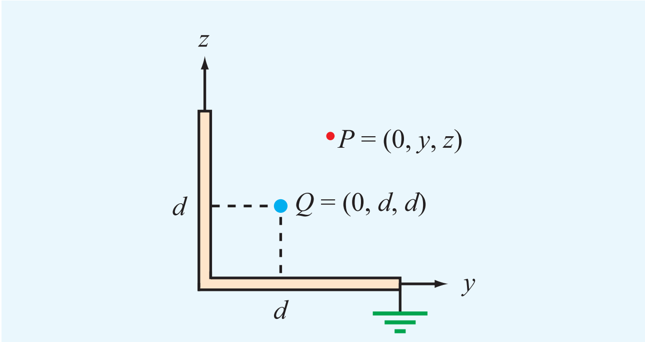

Figure P4.62 Charge Q next to two perpendicular, grounded, conducting half-planes.

Solution:

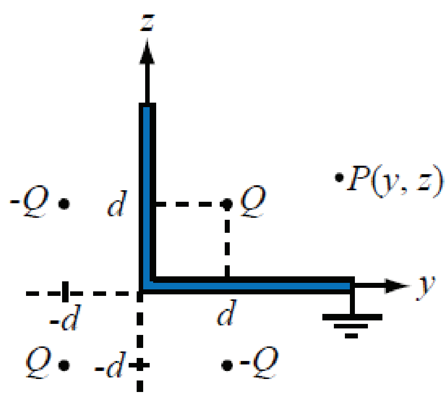

Figure P4.62 (a) Image charges.

(a) The original charge has magnitude and polarity $+Q$ at location $(0, d, d)$. Since the negative y-axis is shielded from the region of interest, there might as well be a conducting half-plane extending in the $-y$ direction as well as the $+y$ direction. This ground plane gives rise to an image charge of magnitude and polarity $-Q$ at location $(0, d, -d)$. In addition, since charges exist on the conducting half plane in the $+z$ direction, an image of this conducting half plane also appears in the $-z$ direction. This ground plane in the x-z plane gives rise to the image charges of $-Q$ at $(0, -d, d)$ and $+Q$ at $(0, -d, -d)$.

(b) Using Eq. (4.47) with $N = 4$,

$$
\begin{aligned}
V(x, y, z) &= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y-d) + \hat{\mathbf{z}}(z-d)|} - \frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y+d) + \hat{\mathbf{z}}(z-d)|}\right. \\
&\quad \left. + \frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y+d) + \hat{\mathbf{z}}(z+d)|} - \frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y-d) + \hat{\mathbf{z}}(z+d)|}\right) \\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2 + (y-d)^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + (y+d)^2 + (z-d)^2}}\right. \\
&\quad \left. + \frac{1}{\sqrt{x^2 + (y+d)^2 + (z+d)^2}} - \frac{1}{\sqrt{x^2 + (y-d)^2 + (z+d)^2}}\right) \\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2 + y^2 - 2yd + z^2 - 2zd + 2d^2}}\right. \\
&\quad - \frac{1}{\sqrt{x^2 + y^2 + 2yd + z^2 - 2zd + 2d^2}} \\
&\quad + \frac{1}{\sqrt{x^2 + y^2 + 2yd + z^2 + 2zd + 2d^2}} \\
&\quad \left. - \frac{1}{\sqrt{x^2 + y^2 - 2yd + z^2 + 2zd + 2d^2}}\right) \qquad \text{(V)}.
\end{aligned}
$$

From Eq. (4.51),

$$
\begin{aligned}
\vec{E} &= -\nabla V \\
&= \frac{Q}{4\pi\varepsilon}\left(\nabla\frac{1}{\sqrt{x^2 + (y-d)^2 + (z-d)^2}} - \nabla\frac{1}{\sqrt{x^2 + (y+d)^2 + (z-d)^2}}\right. \\
&\quad \left. + \nabla\frac{1}{\sqrt{x^2 + (y+d)^2 + (z+d)^2}} - \nabla\frac{1}{\sqrt{x^2 + (y-d)^2 + (z+d)^2}}\right)
\end{aligned}
$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)}{\left(x^2+(y-d)^2+(z-d)^2\right)^{3/2}}-\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)}{\left(x^2+(y+d)^2+(z-d)^2\right)^{3/2}}\right.$$

$$\left.+\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)}{\left(x^2+(y+d)^2+(z+d)^2\right)^{3/2}}-\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)}{\left(x^2+(y-d)^2+(z+d)^2\right)^{3/2}}\right) \qquad \text{(V/m)}.$$