5.18 An infinitely long, thin conducting sheet of width w along the x-direction lies in the x–y plane and carries a current I in the $-y$-direction. Determine the following:

(a) The magnetic field at a point P midway between the edges of the sheet and at a height h above it (Fig. P5.18).

(b) The force per unit length exerted on an infinitely long wire passing through point P and parallel to the sheet if the current through the wire is equal in magnitude but opposite in direction to that carried by the sheet.

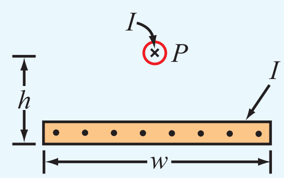

Figure P5.18 A linear current source above a current sheet (Problem 5.18).

Solution:

(a) The sheet can be considered to consist of a large number of infinitely long but narrow wires each dx wide lying next to each other, with each carrying a current $I_x = I\,dx/w$. If we choose the coordinate system shown in Fig. P5.18, the wire at a distance x from the origin is at a distance vector $\mathbf{R}$ from point P, with

$$\mathbf{R} = -\hat{\mathbf{x}}x + \hat{\mathbf{z}}h.$$

Equation (5.30) provides an expression for the magnetic field due to an infinitely long wire carrying a current I as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \hat{\boldsymbol{\phi}}\,\frac{I}{2\pi r}.$$

We now need to adapt this expression to the present situation by replacing I with $I_x = I\,dx/w$, replacing r with $R = (x^2 + h^2)^{1/2}$, and by assigning the proper direction for the magnetic field. From the Biot–Savart law, the direction of $\mathbf{H}$ is governed by $\mathbf{l} \times \mathbf{R}$, where $\mathbf{l}$ is the direction of current flow. In the present case, $\mathbf{l}$ is out of the page, which is the $-\hat{\mathbf{y}}$ direction. Hence, the direction of the field is

$$\frac{\mathbf{l} \times \mathbf{R}}{|\mathbf{l} \times \mathbf{R}|} = \frac{-\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}x + \hat{\mathbf{z}}h)}{|-\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}x + \hat{\mathbf{z}}h)|} = \frac{-(\hat{\mathbf{x}}h + \hat{\mathbf{z}}x)}{(x^2 + h^2)^{1/2}}.$$

Therefore, the field $d\mathbf{H}$ due to current I_x is

$$d\mathbf{H} = \frac{-(\hat{\mathbf{x}}h + \hat{\mathbf{z}}x)}{(x^2 + h^2)^{1/2}}\,\frac{I_x}{2\pi R} = \frac{-(\hat{\mathbf{x}}h + \hat{\mathbf{z}}x)I\,dx}{2\pi w(x^2 + h^2)},$$

and the total field is

$$\begin{aligned}
\vec{H}(0,0,h) &= \int_{x=-w/2}^{w/2} -(\hat{\mathbf{x}}h+\hat{\mathbf{z}}x)\,\frac{I\,dx}{2\pi w\,(x^2+h^2)} \\
&= \frac{-I}{2\pi w}\int_{x=-w/2}^{w/2} (\hat{\mathbf{x}}h+\hat{\mathbf{z}}x)\,\frac{dx}{x^2+h^2} \\
&= \frac{-I}{2\pi w}\left(\hat{\mathbf{x}}h\int_{x=-w/2}^{w/2}\frac{dx}{x^2+h^2}+\hat{\mathbf{z}}\int_{x=-w/2}^{w/2}\frac{x\,dx}{x^2+h^2}\right) \\
&= \frac{-I}{2\pi w}\left(\hat{\mathbf{x}}h\left(\frac{1}{h}\tan^{-1}\left(\frac{x}{h}\right)\right)\Bigg|_{x=-w/2}^{w/2}+\hat{\mathbf{z}}\left(\tfrac{1}{2}\ln\left(x^2+h^2\right)\right)\Big|_{x=-w/2}^{w/2}\right) \\
&= -\hat{\mathbf{x}}\frac{I}{\pi w}\tan^{-1}\left(\frac{w}{2h}\right) \qquad \text{(A/m)}.
\end{aligned}$$

At P in Fig. P5.18, the field is pointing to the left. The z-component could have been assumed zero with a symmetry argument. An alternative solution is to employ Eq. (5.24a) directly.

(b) From Eq. (5.9), a differential force is of the form $d\vec{F}_{\mathrm{m}} = I\,d\vec{l}\times\vec{B}$ or, assuming $d\vec{l} = \hat{\mathbf{a}}_\ell\,d\ell$, the force per unit length is given by

$$\vec{F}'_{\mathrm{m}} = \frac{\partial\vec{F}_{\mathrm{m}}}{\partial\ell} = I\hat{\mathbf{a}}_\ell\times\vec{B} = I\hat{\mathbf{y}}\times\left(-\hat{\mathbf{x}}\frac{\mu_0 I}{\pi w}\tan^{-1}\left(\frac{w}{2h}\right)\right) = \hat{\mathbf{z}}\frac{\mu_0 I^2}{\pi w}\tan^{-1}\left(\frac{w}{2h}\right) \qquad \text{(N)}.$$

The force is repulsive; the wire is experiencing a force pushing it up.