5.25 A cylindrical conductor whose axis is coincident with the z-axis has an internal magnetic field given by

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{2}{r}[1-(4r+1)e^{-4r}] \qquad \text{(A/m)} \qquad \text{for } r \le a$$

where a is the conductor's radius. If $a = 5$ cm, what is the total current flowing in the conductor?

Solution: We can follow either of two possible approaches. The first involves the use of Ampère's law and the second one involves finding $\mathbf{J}$ from $\mathbf{H}$ and then $\mathbf{I}$ from $\mathbf{J}$. We will demonstrate both.

Approach 1: Ampère's law

Applying Ampère's law at $r = a$,

$$\oint_C \mathbf{H}\cdot d\boldsymbol{\ell}\Big|_{r=a} = I$$

$$\int_0^{2\pi} \hat{\boldsymbol{\phi}}\,\frac{2}{r}[1-(4r+1)e^{-4r}]\cdot\hat{\boldsymbol{\phi}}\,r\,d\phi\Bigg|_{r=a} = I$$

$$I = 4\pi[1-(4a+1)e^{-4a}] \qquad \text{(A)}.$$

For $a = 5$ cm, $I = 0.22$ (A).

Approach 2: $\mathbf{H} \rightarrow \mathbf{J} \rightarrow I$

$$\begin{aligned}
\mathbf{J} &= \nabla\times\mathbf{H} \\
&= \hat{\mathbf{z}}\,\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) \\
&= \hat{\mathbf{z}}\,\frac{1}{r}\frac{\partial}{\partial r}(2[1-(4r+1)e^{-4r}]) \\
&= \hat{\mathbf{z}}\,\frac{1}{r}[-8e^{-4r}+8(4r+1)e^{-4r}] \\
&= \hat{\mathbf{z}}\,32e^{-4r}.
\end{aligned}$$

$$\begin{aligned}
I = \int_S \mathbf{J}\cdot d\mathbf{s} &= \int_{r=0}^{a} \hat{\mathbf{z}}\,32e^{-4r}\cdot\hat{\mathbf{z}}\,2\pi r\,dr \\
&= 64\pi\int_{r=0}^{a} re^{-4r}\,dr \\
&= \frac{64\pi}{16}[1-(4a+1)e^{-4a}] \\
&= 4\pi[1-(4a+1)e^{-4a}] \qquad \text{(A)}.
\end{aligned}$$