

5.27 A uniform current density given by

$$\mathbf{J} = \hat{\mathbf{z}}J_0 \quad (\text{A/m}^2)$$

gives rise to a vector magnetic potential

$$\mathbf{A} = -\hat{\mathbf{z}} \frac{\mu_0 J_0}{4} (x^2 + y^2) \quad (\text{Wb/m})$$

- (a) Apply the vector Poisson's equation to confirm the above statement.
- (b) Use the expression for \mathbf{A} to find \mathbf{H} .
- (c) Use the expression for \mathbf{J} in conjunction with Ampère's law to find \mathbf{H} . Compare your result with that obtained in part (b).

Solution:

(a)

$$\begin{aligned} \nabla^2 \mathbf{A} &= \hat{\mathbf{x}} \nabla^2 A_x + \hat{\mathbf{y}} \nabla^2 A_y + \hat{\mathbf{z}} \nabla^2 A_z = \hat{\mathbf{z}} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left[-\mu_0 \frac{J_0}{4} (x^2 + y^2) \right] \\ &= -\hat{\mathbf{z}} \mu_0 \frac{J_0}{4} (2 + 2) = -\hat{\mathbf{z}} \mu_0 J_0. \end{aligned}$$

Hence, $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ is verified.

(b)

$$\begin{aligned} \mathbf{H} &= \frac{1}{\mu_0} \nabla \times \mathbf{A} = \frac{1}{\mu_0} \left[\hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] \\ &= \frac{1}{\mu_0} \left(\hat{\mathbf{x}} \frac{\partial A_z}{\partial y} - \hat{\mathbf{y}} \frac{\partial A_z}{\partial x} \right) \\ &= \frac{1}{\mu_0} \left[\hat{\mathbf{x}} \frac{\partial}{\partial y} \left(-\mu_0 \frac{J_0}{4} (x^2 + y^2) \right) - \hat{\mathbf{y}} \frac{\partial}{\partial x} \left(-\mu_0 \frac{J_0}{4} (x^2 + y^2) \right) \right] \\ &= -\hat{\mathbf{x}} \frac{J_0 y}{2} + \hat{\mathbf{y}} \frac{J_0 x}{2} \quad (\text{A/m}). \end{aligned}$$

(c)

$$\begin{aligned} \oint_C \mathbf{H} \cdot d\mathbf{l} &= I = \int_S \mathbf{J} \cdot d\mathbf{s}, \\ \hat{\phi} H_\phi \cdot \hat{\phi} 2\pi r &= J_0 \cdot \pi r^2, \\ \mathbf{H} &= \hat{\phi} H_\phi = \hat{\phi} J_0 \frac{r}{2}. \end{aligned}$$

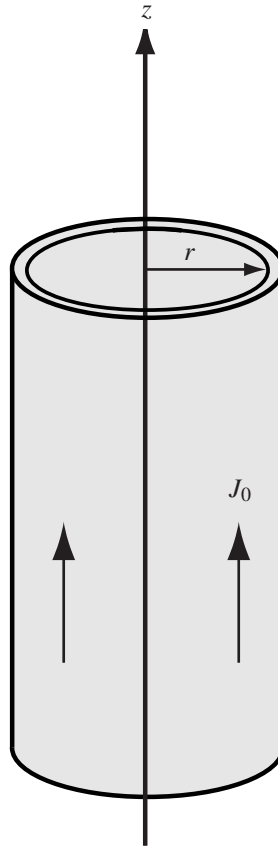


Figure P5.27 Current cylinder of Problem 5.27.

We need to convert the expression from cylindrical to Cartesian coordinates. From Table 3-2,

$$\hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi = -\hat{x} \frac{y}{\sqrt{x^2 + y^2}} + \hat{y} \frac{x}{\sqrt{x^2 + y^2}},$$

$$r = \sqrt{x^2 + y^2}.$$

Hence

$$\mathbf{H} = \left(-\hat{x} \frac{y}{\sqrt{x^2 + y^2}} + \hat{y} \frac{x}{\sqrt{x^2 + y^2}} \right) \cdot \frac{J_0}{2} \sqrt{x^2 + y^2} = -\hat{x} \frac{yJ_0}{2} + \hat{y} \frac{xJ_0}{2},$$

which is identical with the result of part (b).
