6.27 The magnetic field in a given dielectric medium is given by

$$\mathbf{H} = \hat{\mathbf{y}}\, 6\cos 2z \sin(2\times 10^7 t - 0.1x) \qquad \text{(A/m)},$$

where x and z are in meters. Determine:

(a) $\mathbf{E}$,

(b) the displacement current density $\mathbf{J}_\text{d}$, and

(c) the charge density ρ_v.

Solution:

(a)

$$\mathbf{H} = \hat{\mathbf{y}}\, 6\cos 2z \sin(2\times 10^7 t - 0.1x) = \hat{\mathbf{y}}\, 6\cos 2z \cos(2\times 10^7 t - 0.1x - \pi/2),$$

$$\widetilde{\mathbf{H}} = \hat{\mathbf{y}}\, 6\cos 2z\, e^{-j0.1x} e^{-j\pi/2} = -\hat{\mathbf{y}}\, j6\cos 2z\, e^{-j0.1x},$$

$$\widetilde{\mathbf{E}} = \frac{1}{j\omega\varepsilon} \nabla \times \widetilde{\mathbf{H}}$$

$$= \frac{1}{j\omega\varepsilon} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -j6\cos 2z\, e^{-j0.1x} & 0 \end{vmatrix}$$

$$= \frac{1}{j\omega\varepsilon} \left\{ \hat{\mathbf{x}} \left[-\frac{\partial}{\partial z} (-j6\cos 2z\, e^{-j0.1x}) \right] + \hat{\mathbf{z}} \left[\frac{\partial}{\partial x} (-j6\cos 2z\, e^{-j0.1x}) \right] \right\}$$

$$= \hat{\mathbf{x}} \left(-\frac{12}{\omega\varepsilon} \sin 2z\, e^{-j0.1x} \right) + \hat{\mathbf{z}} \left(\frac{j0.6}{\omega\varepsilon} \cos 2z\, e^{-j0.1x} \right).$$

From the given expression for $\mathbf{H}$,

$$\omega = 2\times 10^7 \qquad \text{(rad/s)},$$

$$\beta = 0.1 \qquad \text{(rad/m)}.$$

Hence,

$$u_\text{p} = \frac{\omega}{\beta} = 2\times 10^8 \qquad \text{(m/s)},$$

and

$$\varepsilon_\text{r} = \left(\frac{c}{u_\text{p}}\right)^2 = \left(\frac{3\times 10^8}{2\times 10^8}\right)^2 = 2.25.$$

Using the values for ω and ε, we have

$$\widetilde{\mathbf{E}} = (-\hat{\mathbf{x}}\, 30\sin 2z + \hat{\mathbf{z}}\, j1.5\cos 2z) \times 10^3 e^{-j0.1x} \qquad \text{(V/m)},$$

$$\mathbf{E} = \left[-\hat{\mathbf{x}}\, 30\sin 2z \cos(2\times 10^7 t - 0.1x) - \hat{\mathbf{z}}\, 1.5\cos 2z \sin(2\times 10^7 t - 0.1x) \right] \qquad \text{(kV/m)}.$$

(b)

$$\widetilde{\mathbf{D}} = \varepsilon\widetilde{\mathbf{E}} = \varepsilon_r\varepsilon_0\widetilde{\mathbf{E}} = (-\hat{\mathbf{x}}\,0.6\sin 2z + \hat{\mathbf{z}}\,j0.03\cos 2z) \times 10^{-6} e^{-j0.1x} \qquad (\text{C/m}^2),$$

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t},$$

or

$$\widetilde{\mathbf{J}}_d = j\omega\widetilde{\mathbf{D}} = (-\hat{\mathbf{x}}\,j12\sin 2z - \hat{\mathbf{z}}\,0.6\cos 2z)e^{-j0.1x},$$

$$\begin{aligned}\mathbf{J}_d &= \mathfrak{Re}[\widetilde{\mathbf{J}}_d e^{j\omega t}] \\ &= \left[\hat{\mathbf{x}}\,12\sin 2z\sin(2\times 10^7 t - 0.1x) - \hat{\mathbf{z}}\,0.6\cos 2z\cos(2\times 10^7 t - 0.1x)\right] \qquad (\text{A/m}^2).\end{aligned}$$

(c) We can find ρ_v from

$$\nabla \cdot \mathbf{D} = \rho_v$$

or from

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}.$$

Applying Maxwell's equation,

$$\rho_v = \nabla \cdot \mathbf{D} = \varepsilon\nabla \cdot \mathbf{E} = \varepsilon_r\varepsilon_0\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z}\right)$$

yields

$$\begin{aligned}\rho_v = \varepsilon_r\varepsilon_0 \Bigg\{ &\frac{\partial}{\partial x}\left[-30\sin 2z\cos(2\times 10^7 t - 0.1x)\right] \\ &+ \frac{\partial}{\partial z}\left[-1.5\cos 2z\sin(2\times 10^7 t - 0.1x)\right]\Bigg\} \\ = \varepsilon_r\varepsilon_0 &\left[-3\sin 2z\sin(2\times 10^7 t - 0.1x) + 3\sin 2z\sin(2\times 10^7 t - 0.1x)\right] = 0.\end{aligned}$$