

**6.29** A Hertzian dipole is a short conducting wire carrying an approximately constant current over its length  $l$ . If such a dipole is placed along the  $z$ -axis with its midpoint at the origin, and if the current flowing through it is  $i(t) = I_0 \cos \omega t$ , find the following:

(a) The retarded vector potential  $\tilde{\mathbf{A}}(R, \theta, \phi)$  at an observation point  $Q(R, \theta, \phi)$  in a spherical coordinate system.

(b) The magnetic field phasor  $\tilde{\mathbf{H}}(R, \theta, \phi)$ .

Assume  $l$  to be sufficiently small so that the observation point is approximately equidistant to all points on the dipole; that is, assume  $R' \simeq R$ .

**Solution:**

(a) In phasor form, the current is given by  $\tilde{I} = I_0$ . Explicitly writing the volume integral in Eq. (6.84) as a double integral over the wire cross section and a single integral over its length,

$$\tilde{\mathbf{A}} = \frac{\mu}{4\pi} \int_{-l/2}^{l/2} \iint_s \frac{\tilde{\mathbf{J}}(\vec{R}_i) \exp -jkR'}{R'} ds dz,$$

where  $s$  is the wire cross section. The wire is infinitesimally thin, so that  $R'$  is not a function of  $x$  or  $y$  and the integration over the cross section of the wire applies only to the current density. Recognizing that  $\tilde{\mathbf{J}} = \hat{\mathbf{z}} I_0 / s$ , and employing the relation  $R' \approx R$ ,

$$\tilde{\mathbf{A}} = \hat{\mathbf{z}} \frac{\mu I_0}{4\pi} \int_{-l/2}^{l/2} \frac{\exp -jkR'}{R'} dz \approx \hat{\mathbf{z}} \frac{\mu I_0}{4\pi} \int_{-l/2}^{l/2} \frac{\exp -jkR}{R} dz = \hat{\mathbf{z}} \frac{\mu I_0 l}{4\pi R} \exp -jkR.$$

In spherical coordinates,  $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta$ , and therefore

$$\tilde{\mathbf{A}} = \left( \hat{\mathbf{R}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta \right) \frac{\mu I_0 l}{4\pi R} \exp -jkR.$$

(b) From Eq. (6.85),

$$\begin{aligned} \tilde{\mathbf{H}} &= \frac{1}{\mu} \nabla \times \tilde{\mathbf{A}} = \frac{I_0 l}{4\pi} \nabla \times \left[ \left( \hat{\mathbf{R}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta \right) \frac{\exp -jkR}{R} \right] \\ &= \frac{I_0 l}{4\pi} \hat{\boldsymbol{\phi}} \frac{1}{R} \left( \frac{\partial}{\partial R} (-\sin \theta \exp -jkR) - \frac{\partial}{\partial \theta} \left( \cos \theta \frac{\exp -jkR}{R} \right) \right) \\ &= \hat{\boldsymbol{\phi}} \frac{I_0 l \sin \theta \exp -jkR}{4\pi R} \left( jk + \frac{1}{R} \right). \end{aligned}$$


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